ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, No. 3, 1981

SOME INVARIANTS OF KAKUTANI EQUIVALENCE

BY

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ABSTRACT

We introduce some invariants of Kakutani equivalence and using them we prove that any two distinct cartesian powers of the horocycle flow are inequivalent.

This paper was motivated by J. Feldman's r-entropy [2], A. Katok's discussions in [4, p. 152], and by the proof in [6], showing that the cartesian product of the horocycle flow with itself is not loosely Bernoulli.

Let $T = \{T_t\}$ be a measure preserving flow on a probability space $(X, \mathcal{B}, \mu)$. For $x \in X$ let $x_t = T_t x$ and let $I_t(x)$ denote the orbit interval $[x, x_t]$, $t \geqq 0$. Let $P = \{P_1, \cdots, P_a\}$ be a measurable partition of X. If $x \in P_j$ then j is the P-name of x and we write $P(x) = j$.

DEFINITION 1. For $x, y \in X$, $\varepsilon > 0$ and $t > 0$, $I_t(x)$ and $I_t(y)$ are called (ε, P)-matchable if there exists an increasing map h from $[0, t]$ onto itself s.t. if we denote $A = \{u \in [0, t] \mid P(T_u x) = P(T_{h(u)} y)\}$ then $l(A)/t$ and $l(h(A))/t$ are at least $1 - \varepsilon$ where $l(A)$ denotes the length measure of A. We call h an (ε, P)-match from $I_t(x)$ onto $I_t(y)$.

Set $\bar{f}_t(x, y, P) = \inf\{\varepsilon : I_t(x)$ and $I_t(y)$ are (ε, P)-matchable$\}$. It is clear that

$$\bar{f}_t(x, y, P) \leqq \bar{f}_t(x, z, P) + \bar{f}_t(y, z, P) \qquad \text{for all } x, y, z \in X.$$

We call $B_t(x, \varepsilon, P) = \{y \in X : \bar{f}_t(x, y, P) < \varepsilon\}$ the (t, P)-ball of radius $\varepsilon > 0$ centered at $x \in X$, $t > 0$.

A family $\alpha_t(\varepsilon, P)$ of (t, P)-balls of radius $\varepsilon > 0$ is called an (ε, t, P)-cover of X if $\mu(U\alpha_t(\varepsilon, P)) > 1 - \varepsilon$.

† Partially supported by NSF Grant MCS74–19388.
Received April 17, 1980

Denote $K_t(\varepsilon, P) = \inf |\alpha_t(\varepsilon, P)|$ where $|A|$ denotes the number of elements in A and inf is taken over all (ε, t, P)-covers of X.

Let U denote the family of all nondecreasing functions from R^+ onto itself, converging to ∞, i.e. $u \in U$ iff $0 < u(t) \nearrow \infty$ when $t \to \infty$.

For $u \in U$ we denote

$$\beta(u, \varepsilon, P) = \liminf_{t \to \infty} \frac{\log K_t(\varepsilon, P)}{u(t)}. \tag{1}$$

It is clear that if $\varepsilon_1 \leqq \varepsilon_2$ then $\beta(u, \varepsilon_1, P) \geqq \beta(u, \varepsilon_2, P)$ and if $P_1 \leqq P_2$ then $\beta(u, \varepsilon, P_1) \leqq \beta(u, \varepsilon, P_2)$.

We define

$$e(u, P) = \limsup_{\varepsilon \to 0} \beta(u, \varepsilon, P),$$

$$e(T, u) = \sup_P e(u, P).$$

We prove the following theorems.

THEOREM 1. *A zero-entropy ergodic* m.p. *flow* $T = \{T_t\}$ *is loosely Bernoulli* (LB) (see [1], [4], [7] for definitions) *iff* $e(T, u) = 0$ *for all* $u \in U$.

THEOREM 2. *Let* $T = \{T_t\}$ *be an ergodic* m.p. *flow on* $(X, \mathcal{B}, \mu)$ *and let* $\mathcal{P}_1 \leqq \mathcal{P}_2 \leqq \cdots$ *be an increasing sequence of measurable finite partitions of* X s.t. $\bigvee_{n=1}^{\infty} \mathcal{P}_k$ *generates the* σ*-algebra* $\mathcal{B}$ *(we say that* $\{\mathcal{P}_i, i = 1, 2, \cdots\}$ *generates* $\mathcal{B}$*). Then* $e(T, u) = \sup_m e(u, \mathcal{P}_m)$ *for all* $u \in U$.

THEOREM 3. *Let* $T = \{T_t\}$ *and* $\tilde{T} = \{\tilde{T}_t\}$ *be two ergodic Kakutani equivalent* m.p. *flows on* $(X, \mathcal{B}, \mu)$ *and* $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ *respectively. Then* $e(T, u) = e(\tilde{T}, u)$ *for all* $u \in U$, s.t. $\lim_{t \to \infty} u(at)/u(t) = 1$ *for all* $a \in R^+$.

REMARK. In the definition of $\beta(u, \varepsilon, P)$ in (1) we can take instead of the logarithmic function any $v \in U$, i.e. for $u, v \in U$ we can define

$$\beta(u, v, \varepsilon, P) = \liminf_{t \to \infty} \frac{v(K_t(\varepsilon, P))}{u(t)},$$

$$e(u, v, P) = \limsup_{\varepsilon \to 0} \beta(u, v, \varepsilon, P),$$

$$e(u, v, T) = \sup_P e(u, v, P).$$

Theorems 1, 2, and 3 will still hold. The following theorems explain our choice of the logarithmic function in (1).

THEOREM 4. *Let $h^{(n)} = \{h_t^{(n)} = h_t \times \cdots \times h_t\}$ be the n-times cartesian product of the horocycle flow on the unit tangent bundle of a compact surface of constant negative curvature. Let $u \in U$ be $u(t) = \log t$, $t > 0$. Then*

$$3n - 3 \leqq e(h^{(n)}, u) \leqq 3n - 2.$$

REMARK. Apparently by modifying slightly our proof of Theorem 4 one can show that in fact $e(h^{(n)}, u) = 3(n-1)$.

The following theorem follows from Theorems 3 and 4.

THEOREM 5. *If $m \neq n$ then $h^{(n)}$ and $h^{(m)}$ are not Kakutani equivalent.*

PROOF. Let $n > m$. Then $3n - 3 > 3m - 2$. By Theorem 4 $e(h^{(n)}, u) \neq e(h^{(m)}, u)$ and by Theorem 3 $h^{(n)} \not\sim h^{(m)}$. □

I am grateful to J. Feldman and A. Katok for useful conversations.

1. Proofs of Theorems 1, 2, 3

PROOF OF THEOREM 1. $\{T_t\}$ is LB iff every P is LB, i.e. (see [2]) given $\varepsilon > 0$ there exists $t_0 > 0$ s.t. if $t > t_0$ then $K_t(\varepsilon, P) = 1$. Then $e(T_t, u) = 0$ for all $u \in U$.

Suppose now that $e(T, u) = 0$ for all $u \in U$. Then

$$\beta(u, \varepsilon, P) = 0 \qquad \text{for all } u \in U, \quad \varepsilon > 0, \quad P. \tag{2}$$

Given $\varepsilon > 0$ and P let $\tilde{K}_t = \inf\{K_p(\varepsilon, P) : p \geqq t\}$ and let $u(t) = \log \tilde{K}_t$. Then u is nondecreasing and we claim that $u(t) \not\to \infty$ when $t \to \infty$. Indeed, if $u(t) \to \infty$ then $u \in U$ and we would have $\beta(u, \varepsilon, P) \geqq 1$ which contradicts (2).

So $\tilde{K}_t \not\to \infty$ when $t \to \infty$. Therefore there exist $M > 0$ and $\{t_i\}$, $t_i \to \infty$, $i \to \infty$ s.t.

$$K_{t_i}(\varepsilon, P) < M, \qquad i = 1, 2, \cdots. \tag{3}$$

Using nesting arguments of B. Weiss [7] and of A. Katok and E. Sataev [5] one shows that (3) implies P is LB (see also lemma 1 in [6]). □

PROOF OF THEOREM 2. Let $\varepsilon > 0$ and $P = \{P_1, \cdots, P_a\}$ be given. Let m and $Q = \{Q_1, \cdots, Q_a\}$ be s.t. $Q \leqq \mathcal{P}_m$ and $\sum_{i=1}^{a} \mu(Q_i \Delta P_i) < \varepsilon/10$. Let $F = \bigcup_{i=1}^{a} (P_i \Delta Q_i)$. Since T_t is ergodic there are $t_0 > 0$ and a set $Y \subset X$, $\mu(Y) > 1 - \varepsilon/10$ s.t. if $t \geqq t_0$ and $x \in Y$ then

(4) the relative Lebesgue measure of F on the orbit interval $[x, x_t]$ is less than $\varepsilon/5$.

For $t \geqq t_0$ let α be an $(\varepsilon/10, t, \mathcal{P}_m)$-cover of X. Let $Z = \bigcup \alpha \cap Y$, $\mu(Z) > 1 - \varepsilon/5$. Let $\gamma = \alpha \,|\, Z$ (α restricted on Z), i.e., $x, y \in Z$ belong to the same element of γ iff x and y belong to the same ball of α. Clear that $|\gamma| \leqq |\alpha|$. If x and y belong to the same element of γ then $\bar{f}_t(x, y, \mathcal{P}_m) < \varepsilon/5$. This and (4) imply that $\bar{f}_t(x, y, P) < \varepsilon/2$. This says that $|\gamma|$ many (t, P)-balls of radius $\varepsilon/2$ cover Z.

We have just shown that given $\varepsilon > 0$ and P, there exists $\mathcal{P}_m$ and $t_0 > 0$ s.t. if $t \geqq t_0$ then for every $(\varepsilon/10, t, \mathcal{P}_m)$-cover α there is an $(\varepsilon/2, t, P)$-cover γ s.t. $|\gamma| \leqq |\alpha|$. This implies that if $t \geqq t_0$ then $K_t(\varepsilon/2, P) \leqq K_t(\varepsilon/10, \mathcal{P}_m)$ and

$$\beta(u, \varepsilon/2, P) \leqq \beta(u, \varepsilon/10, \mathcal{P}_m) \leqq e(u, \mathcal{P}_m) \leqq \sup\{e(u, \mathcal{P}_m) : m = 1, 2, \cdots\}.$$

This implies that $e(u, P) \leqq \sup_m e(u, \mathcal{P}_m)$ for all P, which completes the proof. □

Proof of Theorem 3. $\{\tilde{T}\}$ is Kakutani equivalent to $\{T_t\}$ means (see [1], [4], [7]) that there exists a measurable $\tau : X \to R^+$, $\int_X \tau d\mu = a$ s.t. if $v(x, t)$ is defined by

$$\int_0^{v(x,t)} \tau(T_u x) du = t \tag{5}$$

then the flow $S_t(x) = T_{v(x,t)}(x)$ preserves the measure $d\nu = (\tau/a) d\mu$ and is isomorphic to $\{\tilde{T}_t\}$. Cočergin has shown (see [4], p. 120] that τ can be taken s.t.

$$L < \tau(x) < M \qquad \text{for some } L, M > 0 \quad \text{and all } x \in X. \tag{6}$$

We shall show that $e(T, u) = e(S, u)$ for all $u \in U$, s.t. $\lim_{t \to \infty} u(at)/u(t) = 1$ for all $a \in R^+$.

Let $\varepsilon > 0$ and P be given. Let $0 < \delta_0 < \varepsilon$ (δ_0 will be chosen later) be s.t. if $0 < \delta < \delta_0$, then

$$\mu(A) < \delta \quad \text{implies } \nu(A) < \varepsilon, \quad A \in \mathcal{B}. \tag{7}$$

Let $0 < \delta < \delta_0$ be fixed. Since T is ergodic there exist $v_0 > 0$ and $Y \subset X$, $\mu(Y) > 1 - \delta/2$ s.t. if $v \geqq v_0$ and $x \in Y$ then

$$\left| \int_0^v \tau(T_u x) du - av \right| < \delta v.$$

This implies by (5) that if $v(x, t) \geqq v_0$ and $x \in Y$ then

$$|t - av(x, t)| < \delta v(x, t).$$

It follows then from (6) that if $t \geqq t_0 = M v_0$ and $x \in Y$ then

$$|t - av(x, t)| < \delta t / L. \tag{8}$$

Let $t \geqq t_0$ and $\tilde{t} = (t - \delta t/L)/a$. Let α be a $(\delta/2, \tilde{t}, P)$-cover of X for T. Let $Z = Y \cap \bigcup \alpha$. Then $\mu(Z) > 1 - \delta$ and

$$\nu(Z) > 1 - \varepsilon \qquad \text{by (7).} \tag{9}$$

Let $\gamma = \alpha \mid Z$ and let x, y belong to the same element of γ. Then $f_{\tilde{t}}(x, y, P, T) < \delta$.

Let h be a (δ, P)-match from the T-orbit $[x, T_{\tilde{t}}(x)]$ onto the T-orbit $[y, T_{\tilde{t}}(y)]$. Let $A = \{s \in [0, \tilde{t}]/P(T_s x) = P(T_{h(s)} y)\}$ and let $l_T(A)$ be the T-length of A on $[0, \tilde{t}]$. We have

$$l_T(A), l_T(h(A)) > \tilde{t}(1 - \delta). \tag{10}$$

It follows from (8) that the S-orbit $[x, S_t x] = [x, T_{v(x,t)} x] \supset [x, T_{\tilde{t}} x]$ and the S-orbit $[y, S_t y] = [y, T_{v(y,t)} y] \supset [y, T_{\tilde{t}} y]$. We define a map $\tilde{h}$ from $[0, v(x, t)]$ onto $[0, v(y, t)]$ by

$$\tilde{h}(s) = h(s) \quad \text{if } s \in [0, \tilde{t}] \qquad \text{and}$$

$$\tilde{h} \text{ on } [\tilde{t}, v(x, t)] \text{ is any increasing map onto } [\tilde{t}, v(y, t)].$$

It follows from (6), (8), and (10) that

$$\begin{aligned} &l_S[\tilde{t}, v(x, t)], l_S[\tilde{t}, v(y, t)] < 2M\frac{\delta t}{La} \qquad \text{and} \\ &l_S(A), l_S(h(A)) > t(1 - M\delta(1 - \delta/L)). \end{aligned} \tag{11}$$

(11) shows that $\tilde{h}$ defines an $M\delta(1 + 4/La)$-match from the S-orbit $[x, S_t x]$ onto the S-orbit $[y, S_t y]$. Let $0 < \delta_0 < \varepsilon$ in (7) be s.t. if $0 < \delta < \delta_0$ then $M\delta(1 + 4/La) < \varepsilon$.

We have proved that if $\delta < \delta_0$, $t \geqq t_0$ and x, y belong to the same element of $\gamma = \alpha \mid Z$ then x, y belong to a (t, P)-ball of radius $\varepsilon > 0$ for S. Since $|\gamma| \leqq |\alpha|$ this implies that for a fixed $0 < \delta < \delta_0/2$ we have

$$K_t(\varepsilon, P, S) \leqq K_{\tilde{t}}(\delta, P, T), \qquad \tilde{t} = t(1 - \delta/L)/a$$

and since $u \in U$ is nondecreasing

$$\frac{\log K_t(\varepsilon, P, S)}{u(t)} \leqq \frac{\log K_{\tilde{t}}(\delta, P, T)}{u(\tilde{t}a)}. \tag{12}$$

(12) implies that

$$\beta(\varepsilon, P, S) \leqq \beta(\delta, P, T) \leqq e(P, T) \leqq e(T, u), \qquad \text{since } \lim_{t \to \infty} \frac{u(ta)}{u(t)} = 1.$$

Since this is true for all $\varepsilon > 0$ and P we get $e(S,u) \leqq e(T,u)$.

Since the relation $T \sim S$ is symmetric, we get the same way that

$$e(T,u) \leqq e(S,u).$$

This completes the proof. □

2. Proof of Theorem 4

In this section we shall use some notations and definitions from [6].

Let $G = \mathrm{SL}(2,R)$,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let $\Gamma \subset G$ be a discrete subgroup of G s.t. $M = \Gamma/G$ is compact. We assume that $-I \in \Gamma$ and if $A \in \Gamma$, $A \neq I, -I$ then A is hyperbolic, i.e., $|\mathrm{Tr} A| > 2$.

The horocycle flow $h = \{h_t\}$ on M is defined by

$$h_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad g \in G.$$

We shall also consider the flows

$$h_t^*(\Gamma g) = \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The flows $\{h_t\}$, $\{h_t^*\}$ and $\{g_t\}$ preserve the measure μ on M derived from the Haar measure on G.

We shall need the following fact proved in [6].

Let $x, z \in M$ and let $z = h_r^* g_p x$ for some $|p| < \delta$, $|r| < \delta/t$, where $\delta > 0$ is small. Suppose that $[z, h_u z]$ is s-isomorphic to $[x, h_t x]$ (see [6] for the notion of s-isomorphism) and $h_u z = h_b^* g_a h_t x$. Then

$$(13) \qquad |a| < 2\delta, \quad |b| < 2\delta/t, \quad |u - t| < 2\delta t$$

if δ is sufficiently small.

Let $Y^{(n)} = M \times \cdots \times M$ (n times). The flow $h^{(n)} = \{h_t \times \cdots \times h_t\}$ preserves $\nu^{(n)} = \mu \times \cdots \times \mu$ and is ergodic.

Let $x = (x_1, \cdots, x_n) \in Y^{(n)}$, $x_i \in M$, $i = 1, 2, \cdots, n$.

For $t, K > 0$ and $\delta > 0$ (small) we denote $V^{(n)}(x, t, \delta, K) = \{y \in Y^{(n)} \mid y_1 = h_r^* g_p x_1$ for some $|p| < \delta$, $|r| < \delta/t$ and $y_i = h_{c_i} h_{b_i}^* g_{a_i} w_i$, $i = 2, \cdots, n$ for some $|a_i| < K/t$, $|b_i| < K/t^2$, $|c_i| < \delta$ where $w_i = h_r^* g_p x_i$, $i = 2, \cdots, n\}$.

Let ω be a partition of M into u-cylinders of diameter at most $\delta > 0$ (small) (see [6]) and let $\alpha^{(n)} = \omega \times \omega \times \cdots \times \omega$ in $Y^{(n)}$.

LEMMA 1. *Let* $u \in U$ *be* $u(t) = \log t$, $t > 0$. *Then* $e(h^{(n)}, u) \geqq 3(n-1)$.

PROOF. Let $Y = Y^{(2)}$, $\nu = \nu^{(2)}$, $\alpha = \alpha^{(2)}$, $V = V^{(2)}$. It was shown in [6, proof of theorem 1] that if $\varepsilon > 0$ is sufficiently small then there are $t_0 > 0$, a set $F \subset Y$, $\nu(F) > 1 - \varepsilon$ and constants $K, C(\delta) > 0$, $C(\delta) \to 0$, $\delta \to 0$ s.t. if $x \in F$, $t \geqq t_0$ and $v \in B_t(x, \varepsilon, \alpha)$ (the (t, α)-ball of radius ε centered at x, defined in section 1), then

$$h_p^{(2)} v \in V(h_q^{(2)} x, t, C(\delta), K) \qquad \text{for some } 0 \leqq p, q \leqq t.$$

This implies that given $\varepsilon > 0$ (small) there are $t_0 > 0$, a set $F^{(n)} \subset Y^{(n)}$, $\nu^{(n)}(F^{(n)}) > 1 - \varepsilon$ and constants $K, C(\delta) > 0$ as above s.t. if $x \in F^{(n)}$, $t \geqq t_0$ and $v \in B_t(x, \varepsilon, \alpha^{(n)})$ for $h^{(n)}$ then

$$h_p^{(n)} v \in V^{(n)}(h_q^{(n)} x, t, C(\delta), K) \qquad \text{for some } 0 \leqq p, q \leqq t. \tag{14}$$

Denote

$$E^{(n)}(x, t, C, K) = \bigcup_{q=0}^{t} \bigcup_{p=0}^{t} h_{-p}^{(n)} V^{(n)}(h_q^{(n)} x, t, C, K).$$

It follows from (13) that

$$E^{(n)}(x, t, C, K) \subset \bigcup_{s=-2t}^{t} V^{(n)}(h_s^{(n)} x, t, 2C, 2K) = D^{(n)}(x, t, C, K) = D^{(n)}.$$

It is easy to see from the definition of $V^{(n)}$ that

$$Qt^{-3(n-1)} \leqq \nu^{(n)}(D^{(n)}) \leqq Pt^{-3(n-1)} \tag{15}$$

for some constants $Q, P > 0$ depending only on C and K.

We have shown in (14) that if $x \in F^{(n)} \subset Y^{(n)}$, $t \geqq t_0$ and $v \in B_t(x, \varepsilon, \alpha^{(n)})$ then $v \in E^{(n)}(x, t, C(\delta), K)$.

It follows then from (15) that

$$\nu^{(n)} B_t(x, \varepsilon, \alpha^{(n)}) \leqq Pt^{-3(n-1)}, \qquad x \in F^{(n)}. \tag{16}$$

Let $t \geqq t_0$ and let γ be an $(\varepsilon, t, \alpha^{(n)})$-cover of $Y^{(n)}$ (see section 1). It follows from (16) that

$$|\gamma| \geqq \frac{1 - 2\varepsilon}{P} t^{3(n-1)}$$

and hence

$$K_t(\varepsilon, \alpha^{(n)}) \geqq \frac{1 - 2\varepsilon}{P} t^{3(n-1)}$$

and therefore

$$\beta(u,\varepsilon,\alpha^{(n)})=\liminf_{t\to\infty}\frac{\log K_t(\varepsilon,\alpha^{(n)})}{\log t}\geqq 3(n-1).$$

This says that

$$e(h^{(n)},u)\geqq 3(n-1). \qquad \square$$

LEMMA 2. *Given $\varepsilon>0$ there are $t_0>0$, $Z\subset Y^{(n)}$, $\nu^{(n)}(Z)>1-\varepsilon$ and $\rho, K>0$ s.t. if $t\geqq t_0$, $x\in Z$ and $v\in V^{(n)}(x,t,\rho,K)$ then $v\in B_t(x,\varepsilon,\alpha^{(n)})$.*

PROOF. Let $\partial\omega$ denote the union of boundaries of the u-cylinders of ω and let O_γ denote the γ-neighborhood of $\partial\omega$ in M, $\gamma>0$. Let

$$O_\gamma^{(n)}=\{y=(y_1,\cdots,y_n)\in Y^n \mid y_i\in O_\gamma \text{ for some } i=1,2,\cdots,n\}.$$

For a given $\varepsilon>0$ let $0<\gamma<\min\{\delta/10,\varepsilon/10\}$ be so small that $\mu(O_\gamma)<\varepsilon/10n$ (δ is an upper bound for the diameters of u-cylinders in ω). Then

$$\nu^{(n)}(O_\gamma^{(n)})<\varepsilon/10.$$

Since $h^{(n)}$ is ergodic there exist $t_0>0$ and $Z\subset Y^{(n)}$, $\nu^{(n)}Z>1-\varepsilon$ s.t.

$$\text{if } t\geqq t_0 \text{ and } x\in Z \text{ then}$$

$$\text{the relative length of } O_\gamma^{(n)} \text{ on the orbit } [x,h_t^{(n)}x] \text{ is at most } \varepsilon/5. \tag{17}$$

Let $\rho=\gamma/10$ and let $K=\rho/2$. Let $x\in Z$, $t\geqq t_0$ and $v\in V^{(n)}(x,t,\rho,K)$, i.e.

$$v=(v_1,\cdots,v_n),\quad v_1=h_r^*g_px_1,\quad |p|<\rho,\quad |r|<\rho/t$$

and

$$v_i=h_{c_i}h_{b_i}^*g_{a_i}w_i,\quad |a_i|<K/t,\quad |b_i|<K/t^2,\quad |c_i|<\rho,\quad i=2,\cdots,n$$

where $w_i=h_r^*g_px_i$, $i=1,2,\cdots,n$, $w_1=v_1$.

Let $w=(w_1,\cdots,w_n)$. For $s\in[0,t]$ let w_i^s be s.t. $[w_i,w_i^s]$ is s-isomorphic to $[x_i,h_sx_i]$, $i=1,2,\cdots,n$. Then $w_i^s=h_qw_i$ for some q:

$$|q-s|<2\rho s \qquad \text{and all } i=1,2,\cdots,n. \tag{18}$$

It follows from (13) that

$$d(w_i^s,h_sx_i)<2\rho,\qquad i=1,2,\cdots,n, \tag{19}$$

where d denotes the Riemannian metric on the stable foliation W^s in M (see [6]).

(19) implies via our choice of ρ that if $h_s^{(n)}x \notin O_\gamma^{(n)}$ then $h_s^{(n)}x$ and $h_q^{(n)}w$ have the same $\alpha^{(n)}$-names.

It follows also from (13) that

$$d(h_q v_i, h_q w_i) < \rho, \qquad i = 1, 2, \cdots, n$$

and therefore $h_q^{(n)}w$ and $h_q^{(n)}v$ have the same $\alpha^{(n)}$-names by our choice of ρ.

This implies that if $h_s^{(n)}x \notin O_\gamma^{(n)}$ for some $s \in [0, t]$ then $h_s^{(n)}x$ and $h_q^{(n)}v$ have the same $\alpha^{(n)}$-names.

We map s to q to get a match ϕ from the $h^{(n)}$-orbit $[x, h_t^{(n)}x]$ onto the $h^{(n)}$-orbit $[v, h_t^{(n)}v]$.

It follows then from (17), (18), and our choice of γ and ρ that ϕ is an $(\varepsilon, t, \alpha^{(n)})$-match. This proves that $v \in B_t(x, \varepsilon, \alpha^{(n)})$. □

Proof of Theorem 4. If $v \in B_t(x, \varepsilon/2, \alpha^{(n)})$ then $h_p^{(n)}v \in B_t(x, \varepsilon, \alpha^{(n)})$ for all $|p| < \varepsilon/2$. It follows from Lemma 2 that if $x \in Z$, then

$$R = R(x, t, \varepsilon) = \bigcup_{p=-\varepsilon}^{\varepsilon} h_p^{(n)} V^{(n)}(x, t, \rho, K) \subset B_t(x, 2\varepsilon, \alpha^{(n)}). \tag{20}$$

We have

$$\nu^{(n)}R = Dt^{-[3(n-1)+1]} = Dt^{-(3n-2)}$$

for some $D > 0$ depending only on ε and α.

It is clear that we can cover $Y^{(n)}$ by Qt^{3n-2} many R-sets for some $Q > 0$.

This implies via (20) that there is a $(2\varepsilon, t, \alpha^{(n)})$-cover γ of $Y^{(n)}$ s.t. $|\gamma| \leqq Qt^{3n-2}$ and therefore

$$K_t(2\varepsilon, \alpha^{(n)}) \leqq Qt^{3n-2} \qquad \text{and} \tag{21}$$

$$\beta(u, 2\varepsilon, \alpha^{(n)}) = \liminf_{t\to\infty} \frac{\log K_t(2\varepsilon, \alpha^{(n)})}{\log t} \leqq 3n - 2.$$

Since (21) is true for all small $\varepsilon > 0$ we get

$$e(u, \alpha^{(n)}) \leqq 3n - 2. \tag{22}$$

Now let $\omega_1 \leqq \omega_2 \leqq \cdots$ be an increasing sequence of u-partitions of M, generating the Borel σ-algebra in M. Let $\alpha_i^{(n)} = \omega_i \times \cdots \times \omega_i$. Then $\alpha_1^{(n)} \leqq \alpha_2^{(n)} \leqq \cdots$ and $\{\alpha_i^{(n)}, i = 1, 2, \cdots\}$ generates the Borel σ-algebra in $Y^{(n)}$. (22) is true for all $\alpha_i^{(n)}$. By Theorem 2 we get

$$e(h^{(n)}, u) \leqq 3n - 2.$$

This and Lemma 1 complete the proof. □

References

1. J. Feldman, *Non-Bernoulli K-automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), 16–37.

2. J. Feldman, *r-entropy, equipartition, and Ornstein's isomorphism theorem in* $\mathbf{R}^n$, Israel J. Math. **36** (1980), 321–345.

3. J. Feldman and D. Nadler, *Reparametrization of N-flows of zero entropy*, to appear.

4. A. B. Katok, *Monotone equivalence in the ergodic theory*, Izv. Akad. Nauk Math. **41** (1977), 104–157 (Russian).

5. A. B. Katok and E. A. Sataev, *Interval exchange transformations and flows on surfaces are standard*, Mat. Zamet. **20** (1976), 479–488 (Russian).

6. M. Ratner, *The cartesian square of the horocycle flow is not loosely Bernoulli*, Israel J. Math. **34** (1979), 72–96.

7. B. Weiss, *Equivalence of Measure Preserving Transformations*, Lecture Notes, Institute for Advanced Studies, Hebrew University of Jerusalem, 1976.

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